PHYSICAL REVIEW D VOLUME 16, NUMBER 8 15 OCTOBER 1977

Bound states in field theory from confinement and asymptotic behavior*

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(Received 8 July 1977)

A method of successive approximations is developed that allows the calculation of bound states of confined relativistic or nonrelativistic quantum systems from asymptotic behavior and analyticity. The accuracy of the approximation increases rapidly as the asymptotic behavior is given to larger and larger inverse powers of energy. Because of a special scaling property the method is applicable for systems with many bound states, independent of the strength of the coupling. The method is successfully tested in potential theory for the harmonic-oscillator potential. Application to quantum chromodynamics is discussed where the renormalization group in principle permits the determination of the asymptotic behavior.

I. INTRODUCTION

The experimental and theoretical developments in particle physics over the last few years have raised the hope that we may finally have a theory for strong interactions in the framework of quantum chromodynamics (QCD). This exciting prospect brings with it the enormous challenge to derive the predictions of QCD from first principles and compare them quantitatively with experiments. At least two crucial issues must be decided: The general question of quark confinement and the specific question of the particle spectrum of QCD. Both of these are closely related to the long-standing problem of how to calculate the bound states in quantum field theories.

In this note we would like to examine the bound-state problem in field theory from a point of view that emphasizes the asymptotic behavior and confinement. The reasons for adopting this particular approach are really practical ones. The nonlinear operator field equations are, needless to say, extremely difficult to solve. However, ordinary perturbation theory does provide a formal solution with the Green's functions represented as the sums of appropriate sets of Feynman diagrams. The difficulty has always been how to utilize this formal solution when the interaction strength is not weak. With the introduction of renormalization-group techniques, one of the properties of the formal solution that can be deduced is the asymptotic behavior of the Green's functions. The analytic property that will be used is based on confinement. As long as quarks cannot be easily produced, absolute confinement should be an excellent approximation, in which case the discontinuity across the two- (and many-) particle quark-quark or quark-antiquark thresholds vanish. In other words, these branch points are absent.

The idea is then, if we are given the asymptotic behavior of the (two-point) Green's functions and the requirement that they do not have quark-quark and quark-antiquark thresholds, can anything be said about the bound states which are the poles of these Green's functions?

It turns out that we can develop a method of successive approximations that allows one to calculate the poles of these two-point functions with the accuracy increasing step by step as the asymptotic behavior is given to larger and larger inverse powers of energy. Because of a special scaling property, the method is applicable for strong as well as weak coupling strength. The locations of bound states in QCD have not yet been calculated and, therefore, the validity of our approximation cannot be tested there. On the other hand, exact answers are known in potential theory where we can test our approximation scheme. This is an advantage of emphasizing the asymptotic behavior and analyticity. These concepts are equally applicable in relativistic and nonrelativistic theories. We have successfully applied our method to the harmonic-oscillator potential which one might consider as the potential-theory analog of a confined system with linearly rising trajectories and a crude imitation of asymptotic freedom (the potential is flat at small distances).

The initial motivation of this investigation arises from a paper by Migdal[1] wherein he constructs the matrix Padé approximant to the matrix of two-point functions as a method of achieving infrared regularization of QCD. He was able to obtain approximately the leading Regge trajectories of non-strange mesons. Although the approximation method we finally arrive at seems quite different from his, several important ingredients, namely the use of the renormalization-group results, the absence of threshold branch points, and the appearance of Bessel functions, are already contained in his paper.

The plan of the paper is as follows: In Sec. II we develop the method of using the condition of

confinement as a tool to formulate a perturbation theory for the Green's function. In Sec. III we use this method to solve two exactly solvable examples from potential theory by successive approximation. Section IV summarizes our results and discusses the applications to QCD.

II. CALCULATION OF BOUND STATES IN FIELD THEORIES

We proceed to develop our approximation method of calculating the bound states in field theories given the asymptotic behavior of the two-point functions and the absence of the quark-quark and quark-antiquark threshold branch points as a result of confinement. The two-point functions of the local gauge-invariant field O^i are denoted by [1]

$$G^{ij}_{\{\alpha\beta\}}(x-y)=\langle TO^i_{\alpha_1\cdots\alpha_{n'}}O^j_{\beta_1\cdots\beta_{n''}}\rangle . \tag{2.1}$$

The local gauge-invariant fields O^i are constructed from quarks and gluons. They are traceless and symmetric with respect to their Lorentz indices. In case there is operator mixing, linear combinations of O^i must be used.

In momentum space, we write[1,2]

$$G^{ij}_{\{\alpha\beta\}}(p)=\sum_{\nu=0}^{n}\{f_{\alpha_1\beta_1}\cdots f_{\alpha_\nu\beta_\nu}p_{\alpha_{\nu+1}}\cdots p_{\alpha_{n'}}p_{\beta_{n''}}\}\times(p^2)^{(\tau'+\tau'')/2}\Gamma^{ij}(\nu,g^2,t/\mu^2), \tag{2.2}$$

where $n=\min(n',n'')$, $f_{\alpha\beta}=p_\alpha p_\beta-p^2g_{\alpha\beta}$, $t=p^2$, and $\tau'=\Delta'-n'-2$, Δ' being the normal dimension. The symbol $\{\cdots\}$ means symmetrization and subtraction of traces in the Lorentz indices.[3] The function $\Gamma^{ij}(\nu,g^2,t/\mu^2)$ corresponds to the contribution of the states with spin ν; thus n is the maximum angular momentum of these states.

The assumptions about the asymptotic behavior and analytic property can be discussed in terms of G. From now on we shall only display the dependence on n, t, and μ^2, and denote G by $G(n,t,\mu^2)$.

We define $G_B(n,t,\mu^2)$ such that the difference $G-G_B\to 0$ faster than any inverse power of $(-t)$ as $|t|\to\infty$ everywhere *except* along the positive real axis, where G_B will generally have a branch cut. In the rest of this article we will assume G_B to be given by a finite number of terms since that is all that is available in practice. We will see that our approximation method for calculating the bound states improves quite rapidly with the given number of inverse powers.

When there is no confinement $G(n,t,\mu^2)$ *itself* has branch points corresponding to two- and many-particle quark-quark or quark-antiquark thresholds. As a result of confinement, we require that these branch points are absent, and $G(n,t,\mu^2)$ becomes a meromorphic function of t with poles corresponding to (colorless) bound states of the theory.[4]

The asymptotic form of G_B can be obtained in principle from renormalization-group techniques. For theories with a Wilson-Fisher-type fixed point it is of the form

$$G(n,t,\mu^2)\to G_B(n,t,\mu^2)=A(-t)^{n+\tau+\gamma}[1+\text{a decreasing series in powers of } (-t)], \tag{2.3a}$$

where γ is the anomalous dimension at the fixed point and in A we have suppressed the dependence on n, μ^2 (as well as i, j, and g^2). For asymptotically free theories the asymptotic form is

$$G(n,t,\mu^2)\to G_B(n,t,\mu^2)=B(-t)^{n+\tau}[\ln(-t)]^c\times[1+\text{decreasing series in powers of } (-t) \text{ and } \ln t]. \tag{2.3b}$$

To implement the requirement that $G(n,t,\mu^2)$ does not have branch points, we write, similar to the N/D method in S-matrix theory,

$$G(n,t,\mu^2)-G_B(n,t\mu^2)=N/D , \tag{2.4}$$

with D an entire function of t. The dispersion integral for N is

$$N=\frac{1}{\pi}\int_0^\infty\frac{\mathrm{Im}(G-G_B)D}{t'-t}dt'=-\frac{1}{\pi}\int_0^\infty\frac{(\mathrm{Im}G_B)D}{t'-t}dt' . \tag{2.5}$$

Hence

$$G=G_B-\frac{1}{\pi D}\int_0^\infty\frac{(\mathrm{Im}G_B)D}{t'-t}dt' . \tag{2.6}$$

The absence of branch points for $G(n,t,\mu^2)$ is explicit in this representation.

The whole question is how to determine D, whose zeros are the bound states of the theory. At first sight the demand that the difference $G-G_B$ decreases faster than any inverse power of t would not seem to impose any condition on D. For a confined system we expect to have an infinite number of bound states; hence, D has an infinite number of zeros and it will increase faster than any power of t in the cut t plane. Actually, because of the positivity requirement, N must also have an infinite number of zeros interlacing those of D, and therefore will either decrease or increase faster than any power of t. If, in the cut t plane, N were to increase faster than any power of t, then it will cancel the effect of D, in which case N/D and, therefore, $G-G_B$ will not decrease faster than any power of $1/t$, contrary to our requirement. Thus

we must demand that N decrease faster than any power of $1/t$. This argument is heuristic but it is valid in potential theory, as shown in the Appendix, where we find an exponential behavior for N and D, with D increasing and N decreasing.

With N falling off faster than any inverse power of t, the following moment conditions are obtained by expanding (2.5):

$$\int_0^\infty (t')^p \mathrm{Im}(G_B) D\, dt' = 0 \quad \text{for integers } p \geq 0 . \tag{2.7}$$

Thus the demand that $G - G_B$ fall off faster than any inverse power of t will enable us to determine D through the moment conditions (2.7).[5]

Since G_B is given to a finite number of terms, it is not required to satisfy more moment conditions (2.7) for D than is necessary to ensure the correct asymptotic behavior for G up to the appropriate inverse power.

We make the following ansatz for D:

$$D = \sum_{m=0}^{N} a_m (t/\mu^2)^{(-\nu-m)/2} J_{\nu+m}((t/\mu^2)^{1/2}) , \tag{2.8}$$

where $\nu = n + \tau$ for asymptotically free theories and $\nu = n + \tau + \gamma$ for theories with nonzero anomalous dimension. This ansatz is motivated by two considerations. First, any entire function of t/μ^2 can be expressed as the infinite series corresponding to (2.8). Second, the finite series will have as the leading asymptotic term:

$$D \to a_0 (t/\mu^2)^{-\nu/2} J_\nu((t/\mu^2)^{1/2}) . \tag{2.9}$$

This behavior is characteristic of any confined-potential model, such as the simple "bag" models (with $\nu = \ell + \frac{1}{2}$ and $1/\mu = R$, the radius of the bag). If the asymptotic behavior for D is taken as a boundary condition for the relativistic case also where the appropriate choice is $\nu = n + \tau + \gamma$ for (2.3a) and $\nu = n + \tau$ for (2.3b), respectively, then the ansatz (2.8) is unique and we can determine the coefficients a_m from the moment conditions (2.7). This situation is analogous to determining D uniquely from dispersion relations in unconfined problems. To avoid Castillejo-Dalitz-Dyson (CDD) ambiguities there, one specifies the asymptotic behavior of D, which is usually taken as $D \to 1$. For the confined case, D is an entire function of t with infinitely many zeros (bound states). Therefore, the asymptotic condition that $D \to 1$, or for that matter any finite power of t, is inappropriate, because then D will have at most a finite number of zeros. Therefore, we require the asymptotic behavior given by (2.9), and this specifies D unambiguously.

III. EXAMPLES FROM POTENTIAL THEORY

As may be expected, the usual S matrix, which is formulated for the situation in which the interaction vanishes at large distances from the scattering center, is not very useful for the case of a confining potential, where $V(r)$ becomes infinite either at a finite distance (bag) or as r becomes infinite (harmonic oscillator). One simple way of seeing this is to recall

$$S(\nu, k) = D(\nu, -k)/D(\nu, k) , \tag{3.1}$$

where $\nu = j + \frac{1}{2}$ and $D(\nu, k) = D^*(\nu^*, -k^*)$, with the origin of k at threshold. For a confined system $D(\nu, k)$ is analytic in k^2, since there is no threshold. Consequently, $D(\nu, k) = D(\nu, -k)$ and $S(\nu, k) = 1$.

One can show[6] that a natural modification of the S matrix for a confined system is

$$\hat{S}(\nu, k^2) = D(-\nu, k^2)/D(\nu, k^2) . \tag{3.2}$$

The function $\hat{S}$ has many interesting properties, besides the obvious one that bound states are simple poles of $\hat{S}$. The most important for our purpose is its asymptotic behavior. For nonsingular potentials $[r^2 V(r) \to 0 \text{ as } r \to 0]$,

$$\hat{S}(\nu, k^2) \to A(\nu)(-k^2)^\nu (1 + \text{series in inverse powers of } k^2). \tag{3.3}$$

From (3.2) and (3.3), we may consider $\hat{S}(\nu, k^2)$ as a potential-theory analog of $G(\nu, t, \mu^2)$: It has similar asymptotic behavior, and the poles of $\hat{S}$ are bound states of the system.

In order to obtain a feeling of how the method works, we look at two examples from potential theory, both of which are exactly soluble. We then obtain the asymptotic behavior of $\hat{S}$, which by analogy with the previous section we call $\hat{S}_B$. The D function is then reconstructed using the ansatz (2.8) together with the conditions of confinement, as given by the moment conditions (2.7). The zeros of $D(\nu, k^2)$ then can be considered as Regge trajectories and compared to the exact solution.

A. The infinite well

This will be considered as the confinement analog to a free particle. The particle is free out to a distance R_0 and then is totally confined. There is so to speak no potential, but only one parameter R_0 to give a scale to confinement.

With $k^2 = 0$ at the bottom of this naive bag and with $\nu = j + \frac{1}{2}$, the radial solutions to the Schrödinger equation are given by $k^{-\nu} J_\nu(kr)$. Since a bound state corresponds to the vanishing of the wave function at the confinement radius R_0, the D function is just the radial solution with the coordinate r, replaced by the parameter R_0.

We then have for $\hat{S}$, defined by (3.2),

$$\hat{S}(\nu,k^2)=k^{\nu}J_{-\nu}(kR_0)/k^{-\nu}J_{\nu}(kR_0)\,, \tag{3.4}$$

where $\nu=j+\frac{1}{2}$.

In this very specific example the asymptotic form, $\hat{S}_B$, will have only one term:

$$\hat{S}_B(\nu,k)=(-k^2)^{\nu}\,, \tag{3.5}$$

because we can also write for (3.3)

$$\hat{S}(\nu,k^2)=(-k^2)^{\nu}-ik^{2\nu}\sin\pi\nu H_{\nu}^{(1)}(kR_0)/J_{\nu}(kR_0) \tag{3.6}$$

and therefore, $\hat{S}-\hat{S}_B$, with $\hat{S}_B$ given by (3.5), will vanish faster than any inverse power of k^2 in any direction in the physical k^2 plane, as required by the definition of $\hat{S}_B$.

If we now make the ansatz (2.8)

$$D(\nu,k^2)=\sum_{m=0}^{N}a_m(kR)^{-\nu-m}J_{\nu+m}(kR) \tag{3.7}$$

and substitute this into the moment conditions (2.7) (that is if we demand confinement) we find that all moment conditions are satisfied only when all a_m except a_0 vanish.[7] We thus find that if $\hat{S}_B$ is given exactly by $(-k^2)^{\nu}$, then $D(\nu,k^2)$ must be given exactly by

$$D(\nu,k^2)=k^{-\nu}J_{\nu}(kR)\,. \tag{3.8}$$

This, of course, is the exact answer, which is not surprising. However, it has been reconstructed merely from the knowledge of $\hat{S}_B=(-k^2)^{\nu}$. It should be noted here that $\hat{S}_B$ carries no information of the confining radius R_0. Therefore, D as given by (3.7) will fullfill all requirements with any R. To obtain $R=R_0$ we must now match one arbitrary state (or experimental point). Conversely, the correctness of the answer (3.8) is in no way related to the magnitude of R. This property will be seen to carry over to cases where D can be obtained only by successive approximations and the convergence of the approximation will be independent of the strength of the potential.

B. The harmonic oscillator: $V(r)=g^4r^2$

The previous example is somewhat circular. It only serves to show that the ansatz (2.8) is consistent. However, since the leading term of (2.8) was chosen to correspond to the solution of this free-confined vacuum, that example tells us very little of how the procedure will work in a more realistic case.

The harmonic oscillator is another exactly soluble problem, which is very interesting in several respects. It is well known that this potential gives rise to Regge trajectories, linear in k^2 and two units of angular momentum apart, while the infinite-well trajectories are asymptotically linear in k. The infinite-well trajectories also have intercepts ($k^2=0$) at every negative integer of $\nu=j+\frac{1}{2}$, while the harmonic-oscillator trajectories have intercepts only at odd negative integer ν. Most significantly, the potential has no confining radius R_0, although it is absolutely confining, because the classical turnaround points are functions of energy and angular momentum as well as the coupling constant g^2. The parameter R in the ansatz will play a very different role here, and we would expect it to become larger—and, eventually, infinite—with the order of approximation, as the potential gradually will support the linearly rising trajectories.

Because of these differences from the simple bag, the harmonic oscillator is an excellent laboratory for our asymptotics-confinement method.

Considerations similar to the infinite-well case lead to the exact $\hat{S}$ matrix:

$$\hat{S}(\nu,k^2)=(4g^2)^{\nu}\frac{\Gamma(\nu/2+\frac{1}{2}-k^2/4g^2)}{\Gamma(-\nu/2+\frac{1}{2}-k^2/4g^2)} \tag{3.9}$$

For convenience we define a dimensionless $\hat{S}$ by multiplying by $R^{2\nu}$, where R is an arbitrarary length. We then find the asymptotic series $\hat{S}_B(\nu,kR)$ by expanding (3.9). This gives, with a little tedious algebra,

$$\begin{aligned}\hat{S}_B(\nu,kR)&\equiv R^{2\nu}\hat{S}_B(\nu,k)\\&=(-\kappa^2\rho^2)^{\nu}[1-(\rho^4/6)\nu(\nu^2-1)(-\kappa^2\rho^2)^{-2}\\&\quad+(\rho^8/360)\nu(\nu^2-1)(\nu-2)(\nu-3)\\&\quad\times(5\nu+7)(-\kappa^2\rho^2)^{-4}+\cdots],\end{aligned} \tag{3.10}$$

where $\kappa^2=k^2/2g^2$ and $\rho^2=2g^2R^2$ are the scaled variables.

Note that, in this case, we cannot talk of an exact $\hat{S}_B$, but only to what order in g^2 (ρ^2) the asymptotic expansion is carried out.

The moment condition corresponding to (2.7) for relativistic theories is in potential theory

$$\int_0^{\infty}(k^2)^nD(\nu,k^2)\mathrm{Im}\hat{S}_B(\nu,k)dk^2=0 \tag{3.11}$$

for all integers $n\geq 0$.

Because $\hat{S}_B$ is a series of inverse-integral (rather than fractional) powers of k^2, the situation is particularly simple in that at each order of perturbation, we need only a finite number of terms to satisfy *all* the moment equations (3.11). By order of perturbation here, we mean order of g^2 or ρ^2, which makes the infinite well the "unperturbed" problem by comparing the leading term of (3.10) with the exact "bag" $\hat{S}_B$ given by (3.5).

With

$$\hat{S}_n = D_n(-\nu, k^2, R)/D_n(\nu, k^2, R)\,, \tag{3.12}$$

where n labels the order of perturbation, some more algebra gives

(i) zeroth order:

$$D_0(\nu, \kappa^2, \rho) = (\kappa\rho)^{-\nu} J_\nu(\kappa\rho);$$

(ii) first order (to ρ^4):

$$D_1(\mu, \kappa^2, \rho) = D_0 + [(\kappa\rho)^{-\nu}/24]\rho^4[(\kappa\rho)^{-1}J_{\nu+1}(\kappa\rho) + (\nu-1)(\kappa\rho)^{-2}J_{\nu+2}(\kappa\rho)]\,;$$

(iii) second order (to ρ^8):

$$D_2(\nu, \kappa^2, \rho) = D_1 + [(\kappa\rho)^{-\nu}/288]\rho^8[\tfrac{1}{4}(\kappa\rho)^{-2}J_{\nu+2}(\kappa\rho) + \tfrac{1}{10}(5\nu-7)(\kappa\rho)^{-3}J_{\nu+3}(\kappa\rho) + \tfrac{1}{20}(\nu-1)(5\nu-7)(\kappa\rho)^{-4}J_{\nu+4}(\kappa\rho)]\,.$$

The parameter $\rho = \sqrt{2}gR$ is an arbitrary scale parameter, just as $1/\mu$ in the relativistic case. The asymptotic behavior of $\hat{S}_B$ is the same for all R, which can be different for each order of perturbation.

In Fig. 1 we show the Regge trajectories given by the successive approximations D_0, D_1, and D_2. The value of R is adjusted for each order of approximation to "fit" the ground state ($\nu = \frac{1}{2}, \kappa^2 = \frac{3}{2}$). In other words, $\rho_n = \sqrt{2}gR_n$ is determined by the condition $D_n(\nu = \frac{1}{2}, \kappa^2 = \frac{3}{2}, \rho_n) = 0$. We see that the quality of the approximation is independent of the coupling strength g^2. The approximation improves rapidly when $\hat{S}_B$ is given to larger inverse powers of $-k^2$, especially in the "scattering region" ($k^2 < 0$). The zeroth-order approximation D_0 is also the exact solution to the spherical well. There are intercepts at *all* negative integers and the familiar branch points, pairing two trajectories at negative k^2. The exact harmonic-oscillator trajectories are two units of j apart and linear in κ^2. The intercepts are odd negative integers. The next order D_1 not only improves the linearity of the trajectory near the normalization points, but the top trajectory eludes joining the second trajectory and drops almost linearly as it should. There is no "wrong" intercept at $\nu = -2$. The first wrong intercept is at $\nu = -4$. For D_2, the first wrong trajectory intercept is at $\nu = -6$. The top two trajectories now are fairly linear and two units apart for a range of κ^2 ranging from $\kappa^2 = +3$ to $\kappa^2 = -3$. It is now clear that the successive approximations, with an R readjusted for a low physical state at each stage brings about convergence fastest in the region of interest: low-mass physical states and the scattering region.

IV. CONCLUSION AND SUMMARY

We have presented here a method of obtaining the bound states of quark-systems on the basis of asymptotic behavior and confinement. The confinement criterion determines the analytic property of the Green's function G and implies on the basis of the relations (2.4) and (2.6) that (i) D is an entire function of t with an infinite number of zeros (bound states) along the real t axis, (ii) N/D falls off faster than any inverse power of t. The asymptotics of G is contained in G_B which can be obtained, in principle, from the renormalization-group techniques. Applying the above conditions (i) and (ii) on $G - G_B$ we obtain the moment conditions (2.7). Our approximation scheme is such that the more accurately one knows G_B the more accurately one determines the D function from the conditions (2.7). There is an ambiguity in the asymptotic behavior of D, reminiscent of the old CDD ambiguity, which we have resolved on the basis of the asymptotic behavior in the naive bag model [see (2.9)].

We have tested our method in potential theory with confining potentials of the infinite-square-well (naive bag model) and the harmonic-oscillator types, and have found it to work extremely well.

With reference to the potential theory, it is interesting to note that, in addition to confinement, one can also simulate a nonzero anomalous dimension. This can be accomplished by having a potential of the form $V(r) = g^2/r^2$ for which the leading asymptotic form for $\hat{S}$ becomes $(-k^2)^{(\nu^2+g^2)1/2}$ rather than $(-k^2)^\nu$; thus the "anomalous dimension" $\gamma = (\nu^2+g^2)^{1/2} - \nu$. The logarithmic correction to the leading asymptotic behavior in asymptotically free theories is more difficult to simulate in potential theory. It will probably involve energy-dependent potentials. We are pursuing further the potential-theory analogy as a source of guidance and intuition.

The important task is, of course, to use our approximation method to calculate the bound states of QCD. For field theories with nonzero anomalous dimension $\gamma(g_c^2)$, where g_c^2 is the fixed point, we have, using the ansatz (2.8),

$$D_0 = a_0(t/\mu^2)^{-(n+\tau+\gamma)/2} J_{n+\tau+\gamma}((t/\mu^2)^{1/2})$$

for the zeroth-order approximation. For asymp-

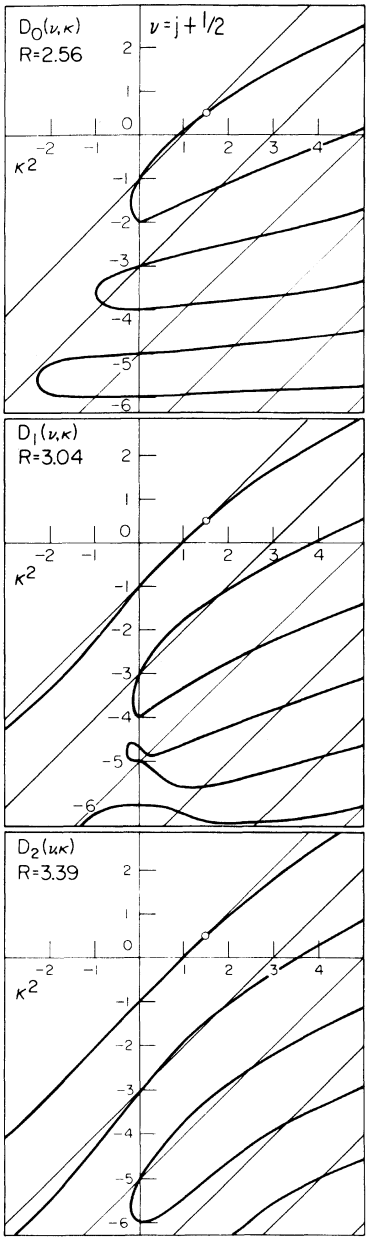

FIG. 1. Successive approximations to the harmonic-oscillator trajectories. Shown are the zeros of $D_0(\nu,\kappa)$, $D_1(\nu,\kappa)$, and $D_2(\nu,\kappa)$, where $\nu=j+\frac{1}{2}$ and $\kappa^2=k^2/2g^2$ (heavy lines). The exact trajectories (thin lines) are two units of j apart. The value of R for each approximation is determined by the "ground" state at $\nu=\frac{1}{2}$ and $\kappa^2=\frac{3}{2}$ in each case. Note the intercepts and "falling" trajectories for D_2.

totically free theories, the zeroth order is given by

$$D_0 = a_0(t/\mu^2)^{-(n+\tau)/2} J_{n+\tau}((t/\mu^2)^{1/2}) .$$

Therefore the leading trajectory here corresponds to the lowest twist.

To obtain a more complete picture of the QCD bound states one must do the following: For a given set of flavor quantum numbers one first finds the anomalous dimension $\gamma(g^2)$ in perturbation theory to a certain order in g^2. Together with the Gell-Mann–Low function $\beta(g^2)$, the renormalization-group equation can then be solved to obtain the leading terms of G_B and a few higher-order terms that are suppressed by powers of $\ln t$.[8] The higher-order terms that are suppressed by powers of t can only be obtained by nonperturbative methods[9]: It remains to be seen how important their effects are. We are currently in the process of carrying out this program.

V. ACKNOWLEDGMENTS

We would like to thank Professor C. H. Woo for many helpful discussions and Dana Beavis for his invaluable help with the numerical calculations. One of us (P.K.) is very grateful to the Aspen Center for Physics for hospitality while part of this work was being carried out. In particular, he is grateful to S. Gasiorowicz for much discussion in the initial stages. One of us (B.R.D.) wishes to thank Dr. Tran Thanh Van for his hospitality at the Université de Paris-Sud, Orsay, France, where part of this work was carried out while on a sabbatical leave.

APPENDIX

Here we present a short potential-theory proof that $N(\nu,k)$ in (2.4) falls faster than any power as $|k| \to \infty$ in the cut complex k plane. This will justify the moment conditions (2.7).

In potential theory, for a system confined to a definite spherical volume [i.e., $V(r)=\infty$, $r>R$], the D function, also called the Jost function, is simply given by[10]

$$D(\nu,k) = \phi_\nu(k, r=R) , \tag{A1}$$

where $\phi_\nu(k,r)$ is the regular wave function defined through the boundary condition at the origin. It satisfies the integral equation[10,11]

$$\phi_\nu(k,r) = k^{-\nu} J_\nu(kr) + \int_0^r dr' g_\nu(k,r,r') V(r') \phi_\nu(k,r') , \tag{A2}$$

where $V(r)$ is the "inside" potential (i.e., for $r<R$). From the above equation one can obtain $\phi_\nu(k,r)$ through iteration—a procedure which is known to converge. The D function satisfies the same integral equation but with r replaced by R.

It can be shown, rigorously, that as $|k| \to \infty$

$$\phi_\nu(k,r) \sim k^{-\nu-1/2} \cos[kr - (\nu+\tfrac{1}{2})\tfrac{1}{2}\pi] \tag{A3}$$

uniformly in r.[10] The right-hand side of (A3) is the

asymptotic form of $k^{-\nu}J_\nu(kr)$, the important point being that in the (convergent) iterated series for $\phi_\nu(k,r)$, the first term $k^{-\nu}J_\nu(kr)$ in (A2) dominates as $|k|\to\infty$. The remaining terms all have similar oscillating behavior but are suppressed by powers of k. There are no terms of finite powers of k without the oscillating functions. Writing the cosine function in the exponential form we obtain, as $|k|\to\infty$,[12]

$$\phi_\nu(k,r)\sim a_\nu(k)e^{ikr}+a_\nu(-k)e^{-ikr}, \tag{A4}$$

where $a(k)=\frac{1}{2}k^{-\nu-1/2}\exp[-i\frac{1}{2}\pi(\nu+\frac{1}{2})]$. From (A1) we then obtain

$$D(\nu,k)\sim a_\nu(k)e^{ikr}+a_\nu(-k)e^{-ikr}. \tag{A5}$$

We remind ourselves that we are considering the limit $|k|\to\infty$ and *not* $r\to\infty$. The value of r is kept fixed at a magnitude less than R.

As $k\to\infty$ along the complex direction, i.e., $k_R\to\infty$ and $k_I\to\infty$ for $k=k_R+ik_I$, we conclude from (A5) that $D(\nu,k)$ has an expoentially *increasing* part $e^{|k_I|R}$ and an exponentially *decreasing* part $e^{-|k_I|R}$. There are no finite powers in k^2 present.

Using the relation

$$\hat{S}=\frac{D(-\nu,k)}{D(\nu,k)}$$

and the asymptotic form (A5), we find that $\hat{S}$ can be expressed as a power series in $1/k^2$, which we have denoted by $\hat{S}_B$, and a part identified as N/D, which *decreases exponentially*,

$$N/D\sim e^{-2|k_I|R}.$$

We note the factor 2 in the exponent above. Since

$$D\sim e^{+|k_I|R}$$

as $|k|\to\infty$, therefore, one must have

$$N\sim e^{-|k_I|R}.$$

We have thus proved that for a confined system, at least within the potential picture, there is, indeed, a term of the N/D type which decreases exponentially and, furthermore, that N itself decreases exponentially with D increasing.

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*Work supported in part by the National Science Foundation.

[1]A. A. Migdal, L. D. Landau Institute for Physics report, 1976 (unpublished).

[2]Our notations closely follow those of Migdal (Ref. 1) and D. J. Gross and F. Wilczek, Phys. Rev. D 9, 980 (1974).

[3]We are grateful to Professor Loyal Durand for some very instructive correspondence on this point.

[4]Our formalism can be modified to accommodate the situation when $G(n,t,\mu^2)$ has branch cuts unrelated to quark-quark and quark-antiquark thresholds: One replaces $G(n,t,\mu^2)$ by

$$G(n,t,\mu^2)-\frac{1}{\pi}\int_C\frac{(\mathrm{Im}G)}{t'-t}\,dt'.$$

[5]These moment conditions are reminiscent of the superconvergence relations.

[6]S.-Y. Chu (unpublished).

[7]When the integral on the left-hand side of (2.7) is not convergent, it is defined by analytic continuation. For example, the integral

$$\int_0^\infty x^\alpha J_\beta(x)dx=2^\alpha\Gamma(\tfrac{1}{2}(\alpha+\beta+1))/\Gamma(\tfrac{1}{2}(\beta-\alpha+1))$$

vanishes for all $\alpha=\beta+2n+1$.

[8]We note that in the presence of the logarithmic terms one can only satisfy a finite number of moment equations (2.7) with a finite number of terms in the ansatz (2.8).

[9]C. G. Callan, Jr., R. F. Dashen, and D. J. Gross, Phys. Lett. 63B, 334 (1976).

[10]For details on this and other relevant aspects of potential theory see R. G. Newton, J. Math. Phys. 1, 319 (1960).

[11]Our $\phi_\nu(k,r)$ is different from Newton's (Ref. 10) by a factor $\frac{1}{2}\sqrt{\pi}\,r$.

[12]This result can also be simply understood from the fact that in the Schrödinger equation,

$$-\frac{d^2\phi_\nu}{dr^2}+V(r)\phi_\nu+\frac{\nu^2-\frac{1}{4}}{r^2}\phi_\nu=k^2\phi_\nu,\quad r\le R$$

as $|k|\to\infty$, the potential term is negligible compared to the k^2 term leading to the asymptotic ϕ_ν of the form $e^{\pm ikr}$ From this follow the results (A4) and (A5).